DE-Vol. 70, Mechanism Synthesis and Analysis
ASME 1994

DESIGNING PLANAR MECHANISMS USING A BI-INVARIANT METRIC IN THE IMAGE SPACE OF *SO* (3)

Pierre Larochelle and J. Michael McCarthy
Department of Mechanical and Aerospace Engineering
University of California, Irvine
Irvine, California

Abstract

In this paper we present a technique for using a bi-invariant metric in the image space of spherical displacements for designing planar mechanisms for $n(>5)$ position rigid body guidance. The goal is to perform the dimensional synthesis of the mechanism such that the distance between the position and orientation of the guided body to each of the n goal positions is minimized. Rather than measure these distances in the plane, we introduce an approximating sphere and identify rotations which are equivalent to the planar displacements to a specified tolerance. We then measure distances between the rigid body and the goal positions using a bi-invariant metric on the image space of $SO(3)$. The optimal linkage is obtained by minimizing this distance over all of the n goal positions.

The paper proceeds as follows. First, we approximate planar rigid body displacements with spherical displacements and show that the error induced by such an approximation is of order $\frac{1}{R^2}$, where R is the radius of the approximating sphere. Second, we use a bi-invariant metric in the image space of spherical displacements to synthesize an optimal spherical $4R$ mechanism. Finally, we identify the planar $4R$ mechanism associated with the optimal spherical solution. The result is a planar $4R$ mechanism that has been optimized for n position rigid body guidance using an approximate bi-invariant metric with an error dependent only upon the radius of the approximating sphere. Numerical results for ten position synthesis of a planar $4R$ mechanism are presented.

1 Introduction

The notion of a metric for measuring distances between two points in a plane, or two points in space, is quite intuitive. However, a metric that measures the distance between finite positions of a rigid body (its location and orientation) in general planar or spatial motion is not. Furthermore, as has been stated by several researchers in the field, there is no bi-invariant metric for planar and spatial motions, see Kazerounian and Rastegar 1992, and Duffy 1990. That is to say, there is no distance measure which is independent of the choice of coordinate system in both the fixed and moving bodies. This distance measure is used to define the goal of our design procedure; it is undesirable to have this goal vary with the choice of coordinates. Therefore, we chose to approximate planar displacements by spherical displacements and use a well known bi-invariant metric in the image space of spherical displacements to design planar $4R$ mechanisms.

Approximate planar motion synthesis, or n position synthesis, in the plane has been studied by Sarkisyan, Gupta and Roth 1973, Gupta and Roth 1975, Suh and Radcliffe 1978, Ravani and Roth 1983, and others. These previous works have involved either the use of metrics which are not bi-invariant, or, have avoided the use of a metric in their procedure which results in: (1) optimization problems which are difficult to solve, and, (2) no measure of how well the resulting linkage meets the design objective of guiding a body through n positions. Our approach involves identifying positions on a sphere which approximate the n desired positions in the plane. We then synthesize an optimal spherical $4R$ mechanism, using a bi-invariant metric, for the spherical positions. Finally, we identify the optimal planar $4R$ mechanism from the spherical solution. The result is a planar $4R$ mechanism which has been optimized for n position rigid body guidance using an approximate bi-invariant metric whose error is dependent only upon the radius of the approximating sphere.

2 The Image Space of Spherical Displacements

First, we review spherical displacements and their representation in the image space. Spherical displacements are a special subset of general spatial displacements in that spherical displacements are pure rotations. Spherical displacements may be represented by a 3×3 orthonormal rotation matrix which describes the orientation of the moving frame relative to the fixed frame. Associated with the matrix of rotation $[A]$ is an axis of rotation $\mathbf{s}$, and a rotation angle about that axis θ, which can be recovered from $[A]$ as follows,

$$\begin{aligned} \theta &= \arccos \frac{a_{11} + a_{22} + a_{33} - 1}{2} \qquad (1) \\ s_x &= \frac{a_{32} - a_{23}}{2 \sin \theta} \\ s_y &= \frac{a_{13} - a_{31}}{2 \sin \theta} \\ s_z &= \frac{a_{21} - a_{12}}{2 \sin \theta} \end{aligned}$$

Using the rotation axis $\mathbf{s}$ and the angle of rotation θ we can represent a spherical displacement by the four dimensional vector $\mathbf{q}$, see Hamilton 1969, which we denote as a quaternion. The four coordinates of the quaternion, sometimes referred to as Euler parameters, are,

$$\begin{aligned} q_1 &= s_x \sin \frac{\theta}{2} \qquad (2) \\ q_2 &= s_y \sin \frac{\theta}{2} \\ q_3 &= s_z \sin \frac{\theta}{2} \\ q_4 &= \cos \frac{\theta}{2} \end{aligned}$$

Note that the components of the quaternion $\mathbf{q}$ satisfy the relation,

$$G_s(\mathbf{q}) \quad : \quad q_1^2 + q_2^2 + q_3^2 + q_4^2 - 1 = 0 \qquad (3)$$

and lie on a unit hypersphere which we denote as *the image space of spherical displacements.*

3 Approximating Planar Displacements

We now examine how spherical displacements may be used to approximate planar displacements with some finite error associated with the radius R of the sphere. The approach used here is similar to the work of McCarthy 1983 and 1986 in which he examined spherical and 3-spherical motions with instantaneous invariants approaching zero and showed that these motions may be identified with planar and spatial motions, respectively.

First, recall that a general planar displacement, (a, b, ψ), in the $z = R$ plane may be described by the coordinate transformation,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [A_p] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (4)$$

where,

$$[A_p] = \begin{bmatrix} \cos\psi & -\sin\psi & a \\ \sin\psi & \cos\psi & b \\ 0 & 0 & R \end{bmatrix} \qquad (5)$$

Now consider a general spherical displacement in which the parameters used to describe the displacement are the three angles θ, ϕ, and ψ as defined in Fig. 1. We refer to θ as the longitude, to ϕ as the latitude, and to ψ as the roll of the position. With this choice of parameters, a general spherical displacement is described by equation,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [A_s] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (6)$$

where,

$$[A_s] = Rot(y, \theta) Rot(x, -\phi) Rot(z, \psi) \qquad (7)$$

Performing the matrix multiplications yields,

$$[A_s] = \begin{bmatrix} c\theta c\psi - s\theta s\phi s\psi & -c\theta s\psi - s\theta s\phi c\psi & s\theta c\phi \\ c\theta s\psi & c\phi c\psi & s\phi \\ -s\theta c\psi - c\theta s\phi s\psi & s\theta s\psi - c\theta s\phi c\psi & c\theta c\phi \end{bmatrix} \qquad (8)$$

where $c\theta = \cos\theta$ and $s\theta = \sin\theta$.

We now define $\hat{a}$ as the longitudinal arc length and $\hat{b}$ as the latitudinal arc length so that, $\hat{a} = R\theta$ and $\hat{b} = R\phi$. Solving for the angles we obtain,

$$\begin{aligned} \theta &= \frac{\hat{a}}{R} \\ \phi &= \frac{\hat{b}}{R} \end{aligned} \tag{9}$$

We now expand the trigonometric functions sine and cosine using a Taylor series about 0,

$$\begin{aligned} \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\ \cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \end{aligned} \tag{10}$$

Substituting the angles θ and ϕ from Eq. 9 into the expansions Eqs. 10 we are able to rewrite Eq. 8 as,

$$[A_s] = \begin{bmatrix} c\psi & -s\psi & \frac{\hat{a}}{R} \\ s\psi & c\psi & \frac{\hat{b}}{R} \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{R}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\hat{a}c\psi - \hat{b}s\psi & \hat{a}s\psi - \hat{b}c\psi & -\frac{1}{2}\left(\frac{\hat{a}^2+\hat{b}^2}{R}\right) \end{bmatrix} + O\left(\frac{1}{R^2}\right) \tag{11}$$

Now, if we consider only the displacement of points $\mathbf{p}$ in the $z = R$ plane, $\mathbf{p} = [x, y, R]^T$ we may we rewrite Eq. 6 using Eq. 11 as,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} c\psi & -s\psi & \hat{a} \\ s\psi & c\psi & \hat{b} \\ 0 & 0 & R \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \frac{1}{R}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\hat{a}c\psi - \hat{b}s\psi & \hat{a}s\psi - \hat{b}c\psi & -\frac{(\hat{a}^2+\hat{b}^2)}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + O\left(\frac{1}{R^2}\right) \tag{12}$$

We note that the first term of Eq. 12 is identical to the equation for general planar displacements, Eq. 4. Moreover, in the limit as $\frac{1}{R} \to 0$ and ($\hat{a}$,$\hat{b}$, x, y) remain finite, spherical and planar motion are identical. Furthermore, we note, to the first order that the spherical motion differs from the motion in the $z = R$ plane only in the z direction.

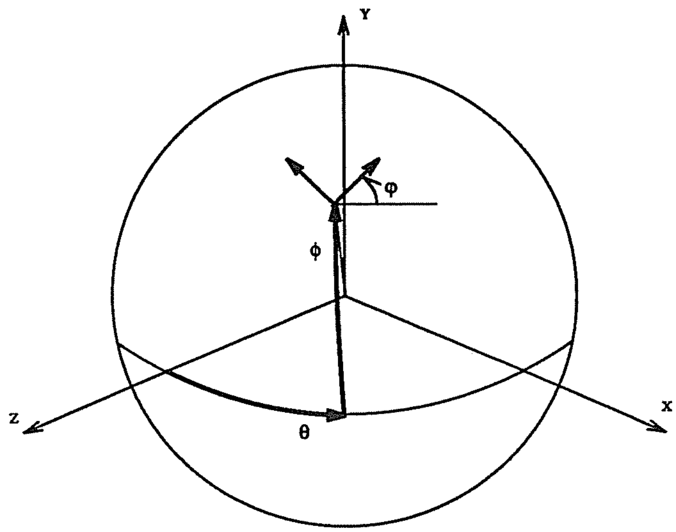

Figure 1: A Spherical Displacement

From our derivation and analysis of Eq. 12 we conclude that spherical displacements may be used to approximate planar displacements with some finite error which is associated with the radius of the sphere. The procedure used to approximate a planar displacement, (a, b, θ), with a displacement on a sphere of radius R is as follows. Examining the first term of Eq. 12 we make the following identifications,

$$\begin{aligned} \hat{a} &\Longleftrightarrow a \\ \hat{b} &\Longleftrightarrow b \\ \psi &\Longleftrightarrow \psi \end{aligned} \tag{13}$$

Finally, using the definition of the arc lengths, Eq. 9, and the radius of the sphere we obtain the three angles; θ, ϕ, and ψ, which describe the spherical displacement on the sphere of radius R that approximates the prescribed planar motion.

$$\begin{aligned} \theta &= \frac{a}{R} \\ \phi &= \frac{b}{R} \\ \psi &= \psi \end{aligned} \tag{14}$$

4 The Metric

The metric in the image space that we use was proposed by Ravani and Roth 1983,1984 for synthesizing planar, spherical and spatial mechanisms for rigid body guidance. Each position of a rigid body maps to a corresponding point in the image space. McCarthy 1990 shows that if the displacement is a general spatial one, the mapping is to the image space of spatial displacements. If the displacement is planar, to the image space of planar displacements, and spherical displacements map to the image space of spherical displacements. Let the two image points of a rigid body displacement be denoted by $\mathbf{q}_1$ and $\mathbf{q}_2$. The measure d of the distance between the two positions is defined as follows,

$$d^2 = (\mathbf{q}_1 - \mathbf{q}_2) \cdot (\mathbf{q}_1 - \mathbf{q}_2) \qquad (15)$$

We now consider a different position of the fixed reference frame described by the image space point $\mathbf{q}_F$. As viewed from the new fixed reference frame the positions are,

$$\begin{aligned} \mathbf{q}_1' &= \mathbf{q}_F \mathbf{q}_1 \\ \mathbf{q}_2' &= \mathbf{q}_F \mathbf{q}_2 \end{aligned} \qquad (16)$$

Next, we compute the distance between the two positions when referenced to the fixed frame at a general position $\mathbf{q}_F$. By substituting Eq. 16 into Eq. 15 we arrive at,

$$d^2 = (\mathbf{q}_F \cdot \mathbf{q}_F)\{(\mathbf{q}_1 - \mathbf{q}_2) \cdot (\mathbf{q}_1 - \mathbf{q}_2)\} \qquad (17)$$

For planar and spatial displacements, $\mathbf{q}_F \cdot \mathbf{q}_F \neq 1$ and the metric is not invariant under a change of fixed frame coordinate system. However in the image space of spherical displacements we have, from Eq. 3, that $\mathbf{q}_F \cdot \mathbf{q}_F = 1$ for all $\mathbf{q}_F$. Therefore, the distance obtained in Eq. 17 is identical to that obtained in Eq. 15 and we conclude that the metric is left invariant. Similarly, suppose $\mathbf{q}_1$ and $\mathbf{q}_2$ are measured with respect to a moving frame. We now examine a change in the position of the moving frame described by the image space point $\mathbf{q}_M$. As viewed from the new moving reference frame the positions are,

$$\begin{aligned} \mathbf{q}_1'' &= \mathbf{q}_1 \mathbf{q}_M \\ \mathbf{q}_2'' &= \mathbf{q}_2 \mathbf{q}_M \end{aligned} \qquad (18)$$

We compute the distance between the two positions by substituting Eq. 18 into Eq. 15 and arrive at,

$$d^2 = \{(\mathbf{q}_1 - \mathbf{q}_2) \cdot (\mathbf{q}_1 - \mathbf{q}_2)\}(\mathbf{q}_M \cdot \mathbf{q}_M) \qquad (19)$$

Again, we have that $\mathbf{q}_M \cdot \mathbf{q}_M = 1$ for all $\mathbf{q}_M$ and conclude that the metric, given by Eq. 15, in the image space of spherical displacements is both left and right invariant, or bi-invariant. That is to say, our measure of the distance between two positions in the image space of spherical displacements is independent of the choice of coordinate system used. For a more general and elaborate discussion of metrics and norms see the chapter on computational kinematics by K.C. Gupta in Erdman, 1992.

5 The Design Procedure

First, determine the spherical positions which approximate the n desired positions of the rigid body by using Eq. 14 and Eq. 8. The next step in the procedure is to synthesize a spherical mechanism which guides a rigid body through the spherical positions which approximate the n desired positions of the moving body in the plane.

The synthesis procedure used to perform the n position spherical $4R$ rigid body guidance problem was presented by Bodduluri and McCarthy 1992, Bodduluri 1990, and Ravani and Roth 1983. The procedure first approximates the point on the constraint manifold of the mechanism which is closest to the desired point in the image space. Then, the metric, given by Eq. 15, is used to measure the distance from this point on the constraint manifold to the desired point in the image space. An optimization problem is then formulated to vary the design parameters of the mechanism such that they minimize this distance for all of the n desired positions. The result is a spherical $4R$ mechanism whose design variables have been optimized such that all of the prescribed positions are either: (1) in the constraint manifold, or, (2) the constraint manifold has been shaped such that it comes as close as possible to all of the desired positions. Once the optimal spherical mechanism has been been found the final step of the design process is to obtain the planar $4R$ mechanism associated with the spherical $4R$ mechanism by using Eq. 14.

6 Case Study

We now present an example of the design of a planar $4R$ mechanism for $n = 10$ positions. The 10 desired positions are listed in Tbl. 1. Based upon the coordinates of the desired positions we choose to limit the design space to a 52×52 square; $(4 \times 13 = 52)$. That is to say we limit the mechanism search space to this planar area such that the moving and fixed pivots, as well as the rigid body being guided, are always within this area. Limiting the angles θ and ϕ about the z axis to; $-15 \leq \theta \leq 15$ and $-15 \leq \phi \leq 15$, we now compute the radius of the approximating sphere,

$$R = \frac{52}{30\frac{\pi}{180}} \approx 100 \qquad (20)$$

The spherical positions which approximate the desired planar positions are found in Tbl. 1. Furthermore, the position error for the optimal spherical $4R$ mechanism is also shown in Tbl. 1. The result of the spherical optimization is the location (longitude and latitude) of the fixed and moving axes of the spherical mechanism. Using Eq. 14, we obtain the locations of the planar axes, listed in Tbl. 2, which are associated with the spherical mechanism. Note that the locations of the fixed axes are given with respect to the fixed frame while the locations of the moving axes are given with respect to the moving reference frame. The length of the fixed and the coupler links are easily found by computing the linear distance between the fixed and moving axes, respectively. To determine the crank lengths of each dyad we use Eq. 14 to recover the linear distance between the fixed and moving axes of the planar linkage from the angular length of the corresponding spherical crank. The four link lengths of the mechanism are listed in Tbl. 4.

In order verify the design we use an approach presented by Suh and Radcliffe 1978 which utilizes the crank constraint equation for a planar RR dyad. We substitute the fixed and moving axes of the mechanism into the crank constraint equation and solve for the crank length. If the mechanism passes exactly through all of the positions then the crank length should remain constant. The results of the verification are presented in Tbl. 3 which lists the standard deviation of the 10 crank lengths obtained from the crank constraint equation. For the sake of comparison we include the results of the verification of the solution presented by Ravani and Roth 1983 to the same 10 positions using the metric given by Eq. 15 in the image space of planar displacements; which you may recall is not bi-invariant. The link lengths of the planar synthesis solution are listed in Tbl. 5. Next we plot the ten desired positions and the coupler curves for both the spherical approximation solution(*solid*) and the planar approximation solution(*dotted*) in Fig. 2. From Fig. 2 it is clear that all of the positions are on one branch of the planar synthesis solution though in Fig. 2 it appears that position 3 is on the second branch of the spherical approximation solution. However, that is not the case. The second branch coupler curve happens to pass through the origin of position 3 however the orientation error here is large. For position 3, while for

Link	Fixed		Moving	
	X	Y	X	Y
DRIVING	13.98	−2.53	10.23	−4.66
DRIVEN	6.57	1.12	2.66	−7.84

Table 2: Mechanism Joint Locations

Link	Length Standard Deviation	
	Sph. Approx.	Planar Syn.
DRIVING	0.560	0.157
DRIVEN	0.288	0.739
	$\sum = 0.8481$	$\sum = 0.8959$

Table 3: Mechanism Verification

the first branch the position error is larger than that for the second branch, the orientation error is much smaller. The approximation to position 3 is on the first branch with the other 9 positions. This confusion arises due to the fact that the orientation of the body is not represented by the coupler curves.

We now illustrate the approximate bi-invariance of our approach by translating the positions −5.0 in both the x and y directions while maintaining their orientations and examine the results obtained for both the planar synthesis method and the spherical approximation method. Using the spherical approximation approach we yield a solution to the translated positions which is nearly identical to the previously determined solution of the untranslated position problem. For the spherical approximation approach, the link lengths for both the original solution and the solution to the translated positions are listed in in Tbl. 4. Using the planar synthesis technique of Ravani and Roth we yield the mechanism with link lengths listed in Tbl. 5. We note that the link lengths have changed for the planar synthesis approach but that the link lengths obtained by using the spherical approximation approach are nearly identical to those found for the original choice of coordinates. For visual comparison, in Fig. 3 we plot the coupler curves to the spherical approximation solution(*solid*) and the planar synthesis solution(*dotted*). Note that the coupler curve for the spherical approximation is the same in both Fig. 2 and in Fig. 3 while the curve for the planar synthesis solution is not.

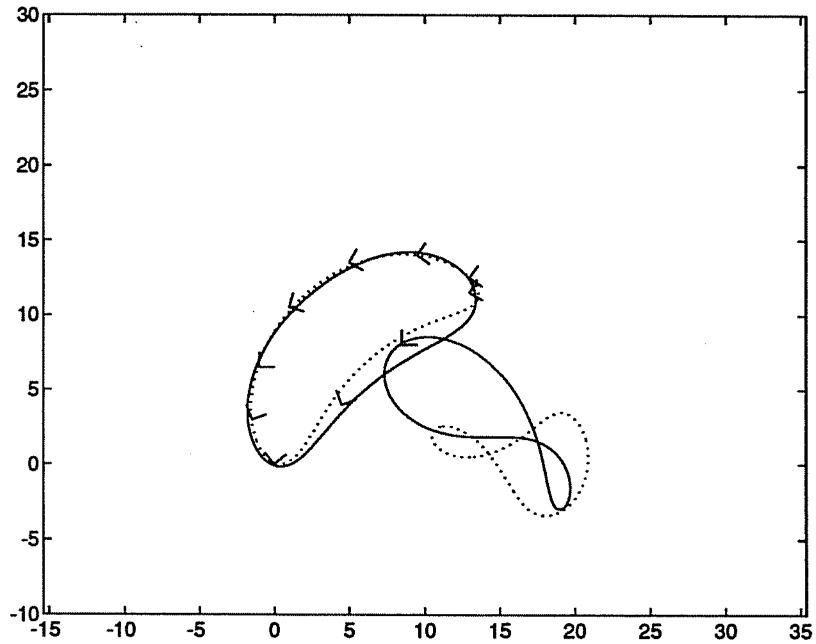

Figure 2: Ravani and Roth's 10 positions with solutions using planar syn.(*dotted*) and sph. approx.(*solid*).

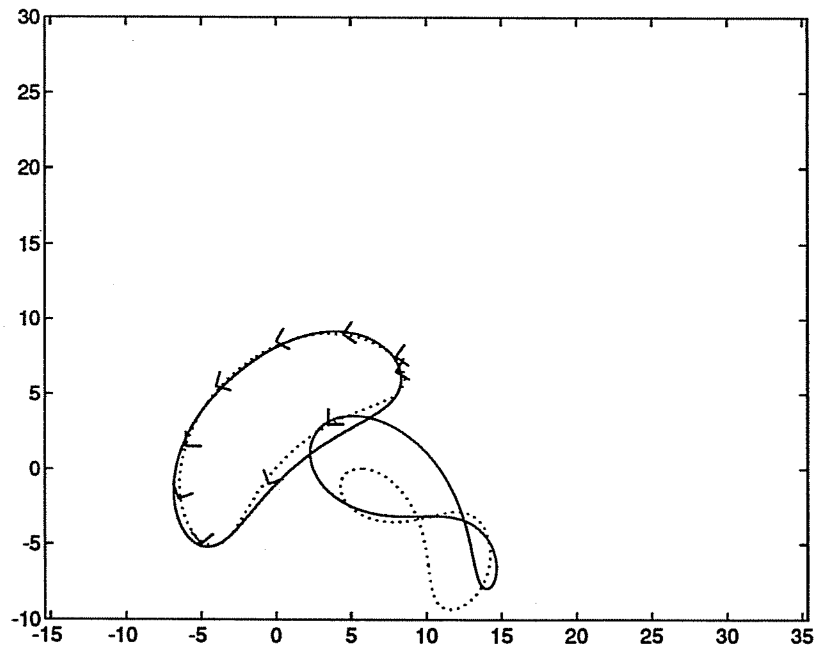

Figure 3: The same 10 positions translated $(-5, -5)$ with solutions using planar syn. and sph. approx.

Pos.	a	b	ψ	Long.	Lat.	Roll	Error
1	0.0	0.0	40.0	0.000	0.000	40.000	$1.07E{-5}$
2	4.5	4.0	20.0	2.578	2.291	20.000	$3.36E{-5}$
3	8.5	8.0	0.0	4.870	4.583	0.000	$1.93E{-5}$
4	13.0	11.5	−30.0	7.448	6.589	−30.000	$1.01E{-5}$
5	13.0	12.5	−35.0	7.448	7.161	−35.000	$2.67E{-5}$
6	9.5	14.0	−35.0	5.443	8.021	−35.000	$3.91E{-6}$
7	5.0	13.5	−30.0	2.864	7.734	−30.000	$6.47E{-6}$
8	1.0	10.5	−15.0	0.572	6.016	−15.000	$6.14E{-6}$
9	−1.0	6.5	0.0	−0.572	3.724	0.000	$1.90E{-5}$
10	−1.5	3.0	20.0	−0.859	1.718	20.000	$1.10E{-1}$
							$\sum = 1.47E{-4}$

Table 1: The 10 Desired Positions

Link	Length(orig)	Length(trans)
DRIVING	6.869	6.864
COUPLER	8.204	8.204
DRIVEN	5.187	5.188
FIXED	8.267	8.277

Table 4: Mechanism Link Parameters: Sph. Approx.

Link	Length(orig)	Length(trans)
DRIVING	5.070	4.802
COUPLER	10.00	8.042
DRIVEN	8.310	7.245
FIXED	7.540	6.772

Table 5: Mechanism Link Parameters: Planar Syn.

7 Conclusion

Recent work in approximate motion synthesis for planar and spatial mechanisms has focused upon devising approximate metrics for planar and spatial displacements, see Kazerounian and Rastegar 1992. These approximate metrics usually involve placing bounds on the operational space of the design and the volume of the body to be guided.

In this paper we have presented our technique of using spherical displacements to approximate planar displacements so that we may synthesize planar mechanisms with a metric that is approximately invariant to choice of coordinate system. We choose the radius of the approximating sphere in accordance with selected bounds on the operational space of the design. Furthermore, we have shown that the error induced by approximating planar motions with spherical motions is of the order $\frac{1}{R^2}$, where R is the radius of the sphere. Moreover, we have demonstrated our technique by synthesizing a planar $4R$ mechanism for a change of coordinates of 10 desired positions.

References

[1] Bodduluri, M., Design and Planned Movement of Multi-Degree of Freedom Spatial Mechanisms. *Ph.D. Dissertation.* University of California, Irvine, 1990.

[2] Bodduluri, M., and McCarthy, J.M., Finite Position Synthesis Using the Image Curve of a Spherical Four Bar Motion. *ASME J. of Mech. Design.* 114:55-60, March 1992.

[3] Bottema, O. and Roth, B., *Theoretical Kinematics.* North-Holland, Amsterdam, 1979.

[4] Duffy, J., Editorial: The Fallacy of Modern Hybrid Control Theory that is Based Upon "Orthogonal Complements" of Twists and Wrench Spaces. *J. of Robotic Systems.* 7(2):139-144, 1990.

[5] Erdman, A., *The First 40 Years of Modern Kinematics.* Wiley, 1992.

[6] Gupta, K.C., and Roth, B., A General Approximation Theory for Mechanism Synthesis. *ASME J. of App. Mech.* 42:451-457, 1975.

[7] Hamilton, W.R., *Elements of Quaternions.* (reprinted) Chelsea Press, 1969.

[8] Kazerounian, K., and Rastegar, J., Object Norms: A Class of Coordinate and Metric Independent Norms for Displacements. *Proc. of the 1992 ASME Design Technical Conferences.* DE-Vol 47:271-275, September 1992.

[9] McCarthy, J.M., Planar and Spherical Rigid Motion as Special Cases of Spherical and 3-Spherical Motion. *ASME J. of Mech., Trans., and Auto. in Design.* 105:569-575, September 1983.

[10] McCarthy, J.M., On the Relation Between Kinematic Mappings of Planar and Spherical Displacements. *ASME J. of App. Mech.* 53:457-459, June 1986.

[11] McCarthy, J.M., *An Introduction to Theoretical Kinematics.* MIT Press, 1990.

[12] Ravani, B., and Roth, B., Motion Synthesis Using Kinematic Mappings. *ASME Journal of Mech., Trans., and Auto. in Design.* 105:460-467, September 1983.

[13] Ravani, B., and Roth, B., Mappings of Spatial Kinematics. *ASME J. of Mech., Trans., and Auto. in Design.* 106:341-347, September 1984.

[14] Sarkisyan, Y.L., Gupta, K.C., and Roth, B., Kinematic Geometry Associated with the Least-Square Approximation of a Given Motion. *ASME J. of Eng. for Ind.* 95:504-510, 1973.

[15] Suh, C.H., and Radcliffe, C.W., *Kinematics and Mechanism Design.* Wiley and Sons, 1978.